Nonlinear Control

Chapter 3: Lyapunov Stability of Autonomous Systems

In this chapter we review the general stability analysis of autonomous nonlinear system, through Laypunov direct and indirect methods, and invariance principles. Furthermore, Lyapunov function generation and Lyapunov-based controller design are reviewed in detail

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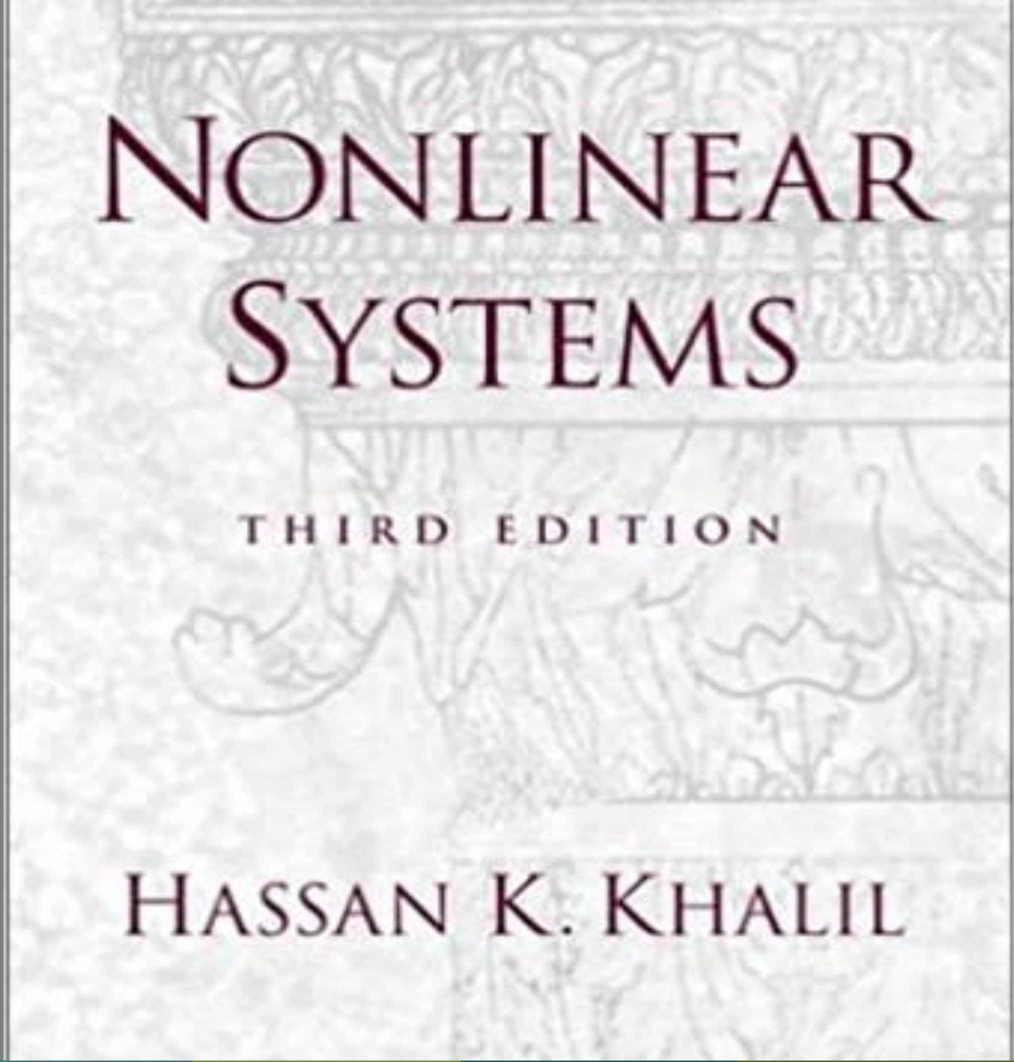
Welcome

To Your Prospect Skills

On Nonlinear System Analysis and

Nonlinear Controller Design . . .



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About ARAS

ARAS Research group originated in 1997 and is proud of its 22+ years of brilliant background, and its contributions to the advancement of academic education and research in the field of Dynamical System Analysis and Control in the robotics application. ***ARAS*** are well represented by the industrial engineers, researchers, and scientific figures graduated from this group, and numerous industrial and R&D projects being conducted in this group. The main asset of our research group is its human resources devoted all their time and effort to the advancement of science and technology. One of our main objectives is to use these potentials to extend our educational and industrial collaborations at both national and international levels. In order to accomplish that, our mission is to enhance the breadth and enrich the quality of our education and research in a dynamic environment.

Contents

In this chapter we review the general stability analysis of autonomous nonlinear system, through Lyapunov direct and indirect methods, and invariance principles. Furthermore, Lyapunov function generation and Lyapunov-based controller design are reviewed in detail. This chapter contains the most important analysis design of nonlinear systems.

Lyapunov Stability of Autonomous Systems

- Definitions
  - ✓ Consider the closed-loop autonomous system

$$\dot{x} = f(x) \qquad (3.1)$$

    - Where $f: D \to R^n$ is a locally Lipschitz map,
    - With eq. point @ origin.

Definition 3.1 *The equilibrium point $x = 0$ of* (3.1) *is*

- *stable if, for each $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ such that*

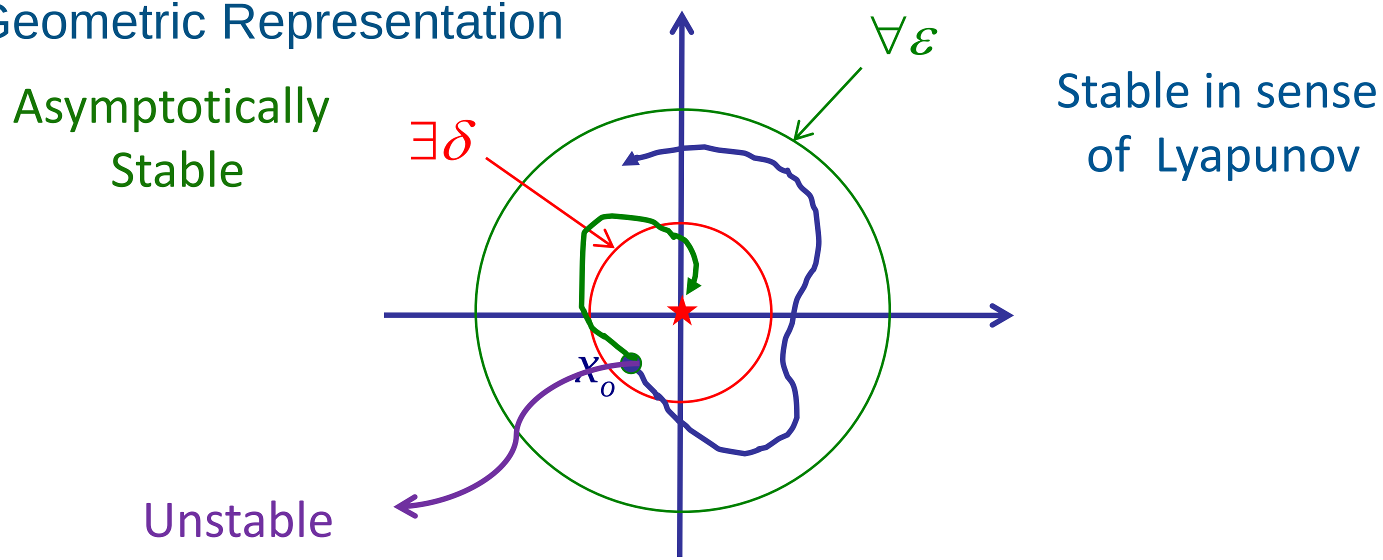
$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \quad \forall\ t \geq 0$$

- *unstable if it is not stable.*
- *asymptotically stable if it is stable and δ can be chosen such that*

$$\|x(0)\| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0$$

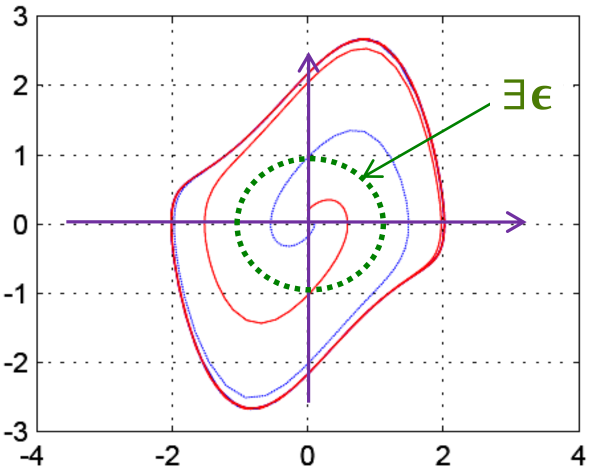
Lyapunov Stability of Autonomous Systems

- Stability Definitions
  - ✓ Stability in sense of Lyapunov:
    - The system trajectory can be kept arbitrary close to the equilibrium point.
  - ✓ Geometric Representation

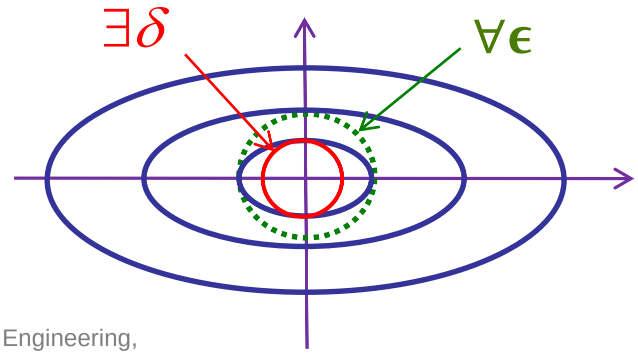


Lyapunov Stability of Autonomous Systems

- Stability Definitions
  - ✓ Example: Van der Pol
    - $\exists \epsilon$ that the trajectories diverges
    - Unstable Eq. Point
    - Stable Limit Cycle



  - ✓ Example: Pendulum
    - $\forall \epsilon \to \exists \delta$ starting from inside δ the trajectory remains in ϵ
    - Stable (not asymptotically)



Lyapunov Stability of Autonomous Systems

- Stability Definitions
  - ✓ Exponential Stability

Definition 3.2 *An equilibrium point* **0** *is exponentially stable if there exist two strictly positive numbers* α *and* λ *such that*

$$\forall t > 0, \quad \| \mathbf{x}(t) \| \leq \alpha \| \mathbf{x}(0) \| e^{-\lambda t} \qquad (3.$$

in some ball $\mathbf{B}_r$ *around the origin.*

  - ✓ Global Stability

Definition 3.3 *If asymptotic (or exponential) stability holds for any initial states, the equilibrium point is said to be asymptotically (or exponentially) stable in the large. It is also called globally asymptotically (or exponentially) stable.*

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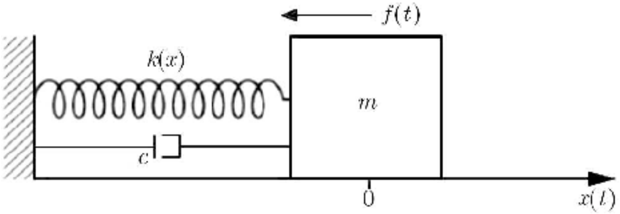
In this chapter we review the general stability analysis of autonomous nonlinear system, through Lyapunov direct and indirect methods, and invariance principles. Furthermore, Lyapunov function generation and Lyapunov-based controller design are reviewed in detail. This chapter contains the most important analysis design of nonlinear systems.

Lyapunov Stability of Autonomous Systems

- Lyapunov Direct Method: The Concept
  - ✓ Mathematical extension of a physical observation:
    - If the total energy is continuously dissipating
    - Then the system (Linear or Nonlinear) must settle down to an equilibrium point.
  - ✓ Example: Mass with nonlinear spring-damper
    - Consider the system:

$$m\ddot{x} + c\dot{x}|\dot{x}| + k_0 + k_1 x^3 = 0$$



- hardening spring + nonlinear damping
  - – Is the resulting motion stable?

Lyapunov Direct Method

- The Concept
  - ✓ Examine the total energy

$$V(x) = \frac{1}{2} m\dot{x}^2 + \int_o^x (k_o x + k_1 x^3)\, dx = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} k_o x^2 + \frac{1}{4} k_1 x^4$$

- Physical observations:
  - – Zero energy corresponds to the equilibrium point $(x = 0, \dot{x} = 0)$
  - – Asymptotic stability implies the convergence of the total energy to zero
  - – Instability is related to the growth of total energy
- Stability is related to the variation of energy

$$\dot{V}(x) = m\dot{x}\ddot{x} + (k_{0x} + k_1 x^3)\dot{x} = \dot{x}(-c\dot{x}|\dot{x}|) = -c|\dot{x}|^3$$

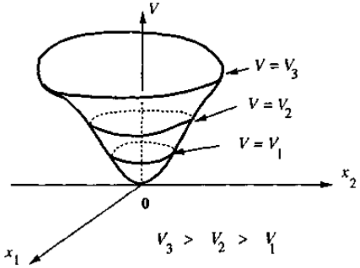
- – The energy of the system is continuously dissipating toward zero
- – The motion is converging to eq. point.

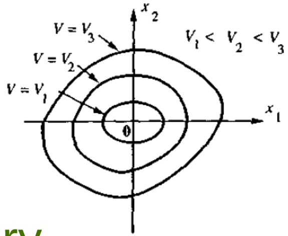
Lyapunov Direct Method

- The Concept
  - ✓ The energy function has three properties:
    - $V(x)$ is a scalar function.
    - $V(x)$ is strictly positive except @ origin.
    - $\dot{V}(x)$ is monotonically decreasing.
  - ✓ In 1892 Lyapunov showed that
    - A certain other function could be used instead of energy
      - Find a positive-definite function $V(x)$ in $D \subset R^n$
      - whose derivative along trajectory is continuously decreasing
      - Then the eq. point is asymptotically stable.

Lyapunov Direct Method

- Direct Method
  - ✓ Positive Definiteness
    - Geometrical Representation
    - Negative Definite
      - – If $-V(x)$ is positive definite
    - Positive Semi-Definite
      - – If $V(0) = 0$ and $V(x) \geq 0$ for $x \neq 0$
    - Time derivative or derivative along trajectory





$$\dot{V}(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}\dot{x}_i = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x)$$

$$= \left[\frac{\partial V}{\partial x_1}, \quad \frac{\partial V}{\partial x_2}, \quad \ldots, \quad \frac{\partial V}{\partial x_n} \right] \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix} = \frac{\partial V}{\partial x} f(x)$$

Lyapunov Direct Method

- Local Stability

Theorem 3.1 *Let $x = 0$ be an equilibrium point for (3.1) and $D \subset R^n$ be a domain containing $x = 0$. Let $V : D \to R$ be a continuously differentiable function such that*

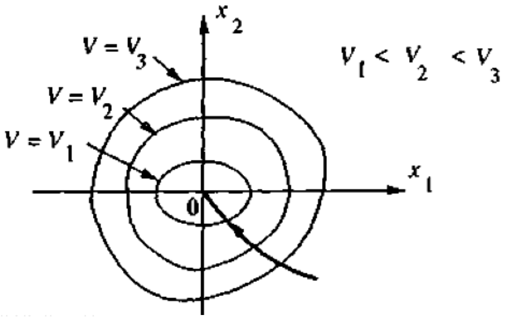
$$V(0) = 0 \text{ and } V(x) > 0 \text{ in } D - \{0\} \tag{3.2}$$

$$\dot{V}(x) \leq 0 \text{ in } D \tag{3.3}$$

Then, $x = 0$ is stable. Moreover, if

$$\dot{V}(x) < 0 \text{ in } D - \{0\} \tag{3.4}$$

then $x = 0$ is asymptotically stable.

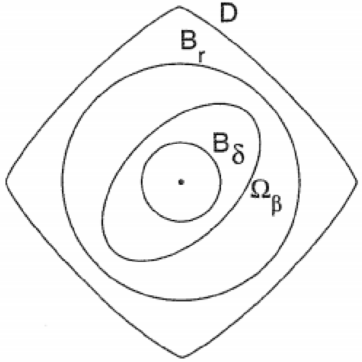


✓ Proof Idea: (Full proof in next slide)

- Lyapunov level Surface: $V(x) = c$ for $c > 0$.
- If $\dot{V}(x) < 0$, then a trajectory crossing a Ly. S., it moves inside and can never come out again.

Lyapunov Direct Method

- Proof
  - ✓ Given $\epsilon > 0$ choose $r \in (0, \epsilon]$, ∋



$$B_r = \{x \in R^n \mid \|x\| \le r\} \subset D$$

Let $\alpha = \min_{||x||=r} V(x)$. Then $\alpha > 0$ by (3.2).

Take $\beta \in (0, \alpha)$, and let

$$\Omega_\beta = \{x \in B_r \mid V(x) \le \beta\}$$

Then, Ω_β is in the interior of B_r. Any trajectory starting from Ω_β at $t = 0$, stays in Ω_β for all $t \ge 0$, because

$$\dot{V}(x(t)) \le 0 \;\Rightarrow\; V(x(t)) \le V(x(0)) \le \beta, \;\forall\, t \ge 0$$

From existence theorem, since Ω_β is a compact set, (3.1) has a unique solution whenever $x(0) \in \Omega_\beta$. As $V(x)$ is continuous and $V(0) = 0$,

$\exists \delta > 0, \ni,$ $\|x\| \le \delta \;\Rightarrow\; V(x) < \beta$ Then $B_\delta \subset \Omega_\beta \subset B_r$

and $x(0) \in B_\delta \Rightarrow x(0) \in \Omega_\beta \Rightarrow x(t) \in \Omega_\beta \Rightarrow x(t) \in B_r$

Lyapunov Direct Method

- Proof (Cont.)

  Therefore $\|x(0)\| < \delta \Rightarrow \|x(t)\| < r \leq \varepsilon, \quad \forall\, t \geq 0$

  which shows that the eq. point is stable. $\square$

  ✓ To show asymptotic stability

  Assume (3.4) holds as well. To show that $\lim_{t\to\infty} x(t) = 0$, one may show that

  $\forall a > 0, \;\; \exists T < 0 \;\; \ni \;\; \big||x(t)|\big| < 0 \quad \forall t > T$.

  Repeat the previous argument for every $a > 0$ we can choose $b > 0$ such that $\Omega_b \subset B_a$. Therefore, it is sufficient to show that $\lim_{t\to\infty} V(x(t)) = 0$.

  Since $V(x(t))$ is monotonically decreasing and bounded from below by zero.

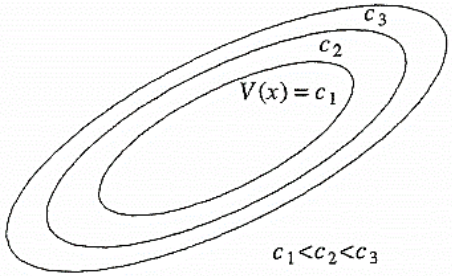
  $$V(x(t)) \to c \geq 0, \text{ as } t \to \infty$$

  To show $c = 0$, use contradiction proof. Suppose $c > 0$. By continuity of $V(x)$ $\exists d > 0 \ni B_d \subset \Omega_c$. The above limit implies that the trajectory $x(t)$ lies outside the ball $B_d, \; \forall t \geq 0$. Let $\gamma = \max_{d \leq ||x|| \leq r} \dot{V}(x)$, By (3.4) $-\gamma < 0$, then

  Since The right hand side will eventually become negative, the inequality contradicts the assumption that $c > 0$.

Lyapunov Direct Method

- Lyapunov function
  - ✓ A Continuously differentiable function $V(x)$ satisfying (3.2) and (3.3) is called a Lyapunov function.
    - The surface $V(x) = c$ for some $c > 0$, is a Lyapunov surface
      - Use the following Lyapunov surfaces:
      - To make the theorem intuitively clear.
      - Upon the condition (3.4)
        - The trajectories crossing a Lyapunov surface moves inside and never come out again.
        - Note if $\dot{V}(x) \leq 0$, then the trajectories may stall!
        - It means it is stable (not going outside)
        - But not necessarily asymptotic stable.
      - If $\dot{V}(x) < 0$, then the Lyapunov surfaces will shrink to origin. Implying asymptotic convergence.



Lyapunov Direct Method

- Example 1: Pendulum without friction
  - System:
$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\left(\frac{g}{l}\right)\sin x_1\end{aligned}$$
  - Lyapunov Candidate: $V(x) = \left(\frac{g}{l}\right)(1-\cos x_1) + \frac{1}{2}x_2^2$
    - How??! (Total Energy)
    - It is positive definite in the domain $-2\pi < x_1 < 2\pi$
  - Lyapunov Function?
    - Derivative along trajectory:
$$\dot{V}(x) = \left(\frac{g}{l}\right)\dot{x}_1\sin x_1 + x_2\dot{x}_2 = \left(\frac{g}{l}\right)x_2\sin x_1 - \left(\frac{g}{l}\right)x_2\sin x_1 = 0$$
    - Eq. point is stable.
    - But not asymptotically stable!
    - Trajectory starting @ Ly. S. $V(x) = c$, remain on it.

Lyapunov Direct Method

- Example 2:
  - ✓ Pendulum with viscous friction
    - System:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\left(\frac{g}{l}\right)\sin x_1 - \left(\frac{k}{m}\right)x_2$$

    - Lyapunov Candidate:
      – The same as Ex1. (Total Energy)

$$V(x) = \left(\frac{g}{l}\right)(1-\cos x_1) + \frac{1}{2}x_2^2$$

    - Lyapunov Function?
      – Derivative along trajectory:

$$\dot{V}(x) = \left(\frac{g}{l}\right)\dot{x}_1 \sin x_1 + x_2\dot{x}_2 = -\left(\frac{k}{m}\right)x_2^2$$

      – Positive Semi-definite: zero irrespective of x_1
      – Only stable but not asymptotically stable!
      – Phase portrait and linearization method → Asy. Stable.
  - ✓ Lyapunov direct conditions are only sufficient!

Lyapunov Direct Method

- Example 2:
  - ✓ Pendulum with viscous friction
    - Use another Lyapunov Candidate:

$$V(x) = \tfrac{1}{2}x^T P x + \left(\frac{g}{l}\right)(1-\cos x_1)$$
$$= \tfrac{1}{2}[x_1 \ x_2]\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22}\end{bmatrix}\begin{bmatrix} x_1 \\ x_2\end{bmatrix} + \left(\frac{g}{l}\right)(1-\cos x_1)$$

    - Lyapunov Function?
      - $V(x) > 0$ if

$$p_{11} > 0;\ p_{22} > 0;\ p_{11}p_{22} - p_{12}^2 > 0$$

    - Derivative along trajectory:

$$\dot{V}(x) = \left(\frac{g}{l}\right)(1-p_{22})x_2 \sin x_1 - \left(\frac{g}{l}\right)p_{12}x_1 \sin x_1 + \left[p_{11} - p_{12}\left(\frac{k}{m}\right)\right]x_1 x_2 + \left[p_{12} - p_{22}\left(\frac{k}{m}\right)\right]x_2^2$$

      - If $p_{12} = 0.5\, k/m$, then

$$\dot{V}(x) = -\tfrac{1}{2}\left(\frac{g}{l}\right)\left(\frac{k}{m}\right)x_1 \sin x_1 - \tfrac{1}{2}\left(\frac{k}{m}\right)x_2^2$$

becomes neg-def. over the domain $D = \{x \in R^2 \mid |x_1| < \pi\}$

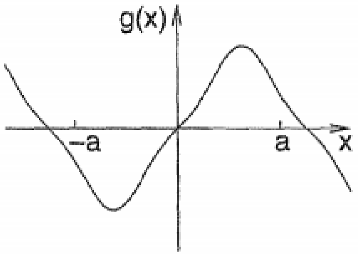
Lyapunov Direct Method

- Example 3:
  - ✓ Consider the general 1st order system

$$\dot{x} = -g(x)$$

Where $g(x)$ is locally Lipschitz on $(-a, a)$, and satisfies:

$$g(0) = 0; \quad xg(x) > 0, \quad \forall\, x \neq 0 \ \text{ and } \ x \in (-a, a)$$



Like in figure. Lyapunov Candidate: $V(x) = \int_0^x g(y)\, dy$

– How??! (Total Energy)

– It is positive definite in the domain $D = (-a, a)$.

- Lyapunov Function?

– Derivative along trajectory:

$$\dot{V}(x) = \frac{\partial V}{\partial x}[-g(x)] = -g^2(x) < 0, \ \forall\, x \in D - \{0\}$$

– The eq. point is Asymptotically stable.

Lyapunov Direct Method

- Example 4:
  - ✓ Consider the following system:

$$\dot{x}_1 = x_1(x_1^2 + x_2^2 - 2) - 4x_1x_2^2$$

$$\dot{x}_2 = 4x_1^2x_2 + x_2(x_1^2 + x_2^2 - 2)$$

- The eq. point is @ origin.
- Lyapunov Candidate: $V(x_1, x_2) = x_1^2 + x_2^2$
  - – Derivative along trajectory:

$$\dot{V} = 2(x_1^2 + x_2^2)(x_1^2 + x_2^2 - 2)$$

  - – It is negative definite in a ball: $x_1^2 + x_2^2 < 2$
  - – The eq. point is asymptotically stable.

Aleksandr Mikhailovich Lyapunov

(June 6 1857 – November 3, 1918)

Was a Russian mathematician, and physicist. He was the son of an astronomer. Lyapunov is known for his development of the stability theory of a dynamical system, as well as for his many contributions to mathematical physics and probability theory.

He studied at the University of Saint Petersburg. In 1880 Lyapunov received a gold medal for a work on hydrostatics. Lyapunov's impact was significant, and a number of different mathematical concepts therefore bear his name: Lyapunov equation, Lyapunov exponent, Lyapunov function, Lyapunov fractal, Lyapunov stability, Lyapunov's central limit theorem, and Lyapunov vector.

By the end of June 1917, Lyapunov traveled with his wife to his brother's place in Odessa. Lyapunov's wife was suffering from tuberculosis so they moved following her doctor's orders. She died on October 31, 1918. The same day, Lyapunov shot himself in the head, and three days later he died.

Lyapunov Direct Method

- Global Stability
  - ✓ If the origin is asymptotically stable
    - Define Region of Attraction (RoA)
      - Let $\phi(t;x)$ be the solution for (3.1), Then RoA is the set of all points x such that $\phi(t;x)$ is defined, and $\forall t \geq 0, \lim_{t\to\infty} \phi(t;x) = 0.$
    - Analytic determination of RoA is hard or even impossible.
    - Lyapunov functions may be used to find an estimate of RoA.
      - Assume a Ly. Function is negative definite in a domain D.
      - Assume $\Omega_c = \{x \in R^n | V(x) \leq c\}$ is bounded and contained in D.
      - Every trajectory starting in Ω_c converges to origin.
      - Ω_c is a (conservative) estimate of the RoA.
  - ✓ If RoA is R^n then eq. point is globally asymptotically stable.

Lyapunov Direct Method

- ## Global Stability (Barbashin-Krasovshii)

Theorem 3.2 *Let $x = 0$ be an equilibrium point for (3.1). Let $V : R^n \to R$ be a continuously differentiable function such that*

$$V(0) = 0 \text{ and } V(x) > 0, \quad \forall\, x \neq 0$$

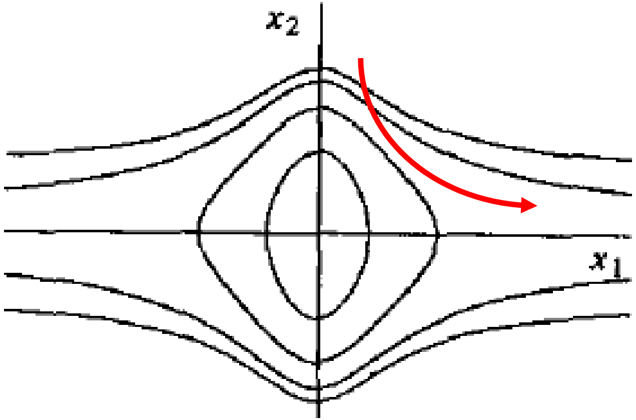
Radially Unbounded

$$\|x\| \to \infty \Rightarrow V(x) \to \infty$$

$$\dot{V}(x) < 0, \quad \forall\, x \neq 0$$

then $x = 0$ is globally asymptotically stable.

- For small c the Ly. Surfaces $V(x) = c$ are closed, but for large c the Ly. S. are not closed, then the trajectory may diverge.



Radial Unboundedness

Lyapunov Direct Method

- Example 5:
  - ✓ System as in Ex3: $\dot{x} = -g(x)$

    In which, $g(0) = 0$ and $xg(x) > 0$ for $x \neq 0$

    - Lyapunov Candidate: $V(x) = x^2$
      - It is positive definite in the whole space
      - It is radially unbounded
    - Lyapunov Function?
      - Derivative along trajectory: $\dot{V} = 2x\dot{x} = -2xg(x)$
      - Hence, $\dot{V} < 0$ as long as $x \neq 0$.
      - Hence, the origin is globally asymptotically stable.
    - Typical Examples
      - $\dot{x} = -x^3$ OR $\dot{x} = \sin^2 x - x, \quad (\sin^2 x \leq |\sin x| < |x|)$
      - $xg(x) = x^4 > 0$ and $xg(x) = x^2 - x\sin^2 x > x^2 - x\,|x| > 0$

Lyapunov Direct Method

- Example 6:
  - ✓ Consider the following system:

$$\dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2)$$

$$\dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2)$$

- The eq. point is @ origin.
- Lyapunov Candidate:

$$V(\mathbf{x}) = x_1^2 + x_2^2$$

  - – Derivative along trajectory:

$$\dot{V}(\mathbf{x}) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = -2(x_1^2 + x_2^2)^2$$

  - – It is negative definite everywhere,
  - – It is radially unbounded,
  - – The eq. point is globally asymptotically stable.

Lyapunov Direct Method

- Some Remarks:

1. Use total energy as the first Lyapunov candidate, but don't limit yourself to that.

2. Many Lyapunov functions exist for a system. If $V(x)$ is a Lyapunov function, so is $V_1 = \rho V^{\alpha}$.   .

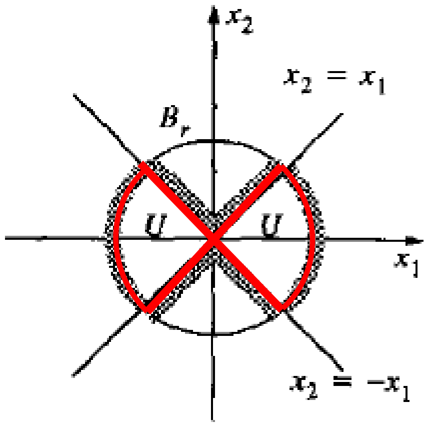
3. Lyapunov theorems are sufficient theorems, if a Lyapunov candidate doesn't work, search for another one!

Lyapunov Direct Method

- Instability Theorem: (Chetaev)
  - ✓ Let $V: D \to R$ be a continuously differentiable function on Domain $D \subset R^n$.
  - ✓ Let D contain the origin, and $V(0) = 0$.
  - ✓ For a point x_0 arbitrary close to origin $V(x_0) > 0$.
  - ✓ Choose $r > 0$ such that the ball $B_r = \{x \in R^n \,||\, ||x|| \leq r\}$ is contained in D.
  - ✓ Let U be a nonempty set in Br, such that

$$U = \{x \in B_r \mid V(x) > 0\} \tag{3.8}$$

  - ✓ Its boundary is the surface $V(x) = 0$.
  - ✓ Example: $V(x) = \frac{1}{2}(x_1^2 - x_2^2)$
    - $V(0) = 0, V(x_0) > 0$ in the hatched area:
    - $V(x) = 0$, at the boundaries of $x_1 = \pm x_2$
    - The region U is the hatched area



Lyapunov Direct Method

- Instability Theorem: (Chetaev)

Theorem 3.3 *Let $x = 0$ be an equilibrium point for (3.1). Let $V : D \to R$ be a continuously differentiable function such that $V(0) = 0$ and $V(x_0) > 0$ for some x_0 with arbitrarily small $\|x_0\|$. Define a set U as in (3.8) and suppose that $\dot{V}(x) > 0$ in U. Then, $x = 0$ is unstable.* ◇

✓ Proof: As long as $x(t)$ is inside U, $V\big(x(t)\big) \geq a$, since $\dot{V}(x) > 0$.

- Let $\gamma = \min\{\dot{V}(x) \mid x \in U \text{ and } V(x) \geq a\}$
- Then $\gamma > 0$ and

$$V(x(t)) = V(x_0) + \int_0^t \dot{V}(x(s))\, ds \geq a + \int_0^t \gamma\, ds = a + \gamma t$$

- This shows that $x(t)$ have to leave U, because $V(x)$ is bounded on U.
- $x(t)$ cannot leave from the surface boundary since $\mathrm{V}(x) = 0$ while $V\big(x(t)\big) \geq a$.
- Therefore it shall leave from the sphere $\big||x|\big| = r$.

Lyapunov Direct Method

- **Instability Theorem:** (Chetaev)
  - ✓ Consider the system:
$$\begin{aligned}\dot{x}_1 &= x_1 + g_1(x)\\ \dot{x}_2 &= -x_2 + g_2(x)\end{aligned}$$
    - Where g_1, g_2 are locally Lipschitz, and satisfy the following in a region D.
$$|g_1(x)| \le k\|x\|_2^2, \qquad |g_2(x)| \le k\|x\|_2^2$$
    - These imply that $g_1(0) = g_2(0) = 0$. Hence the origin is the eq. point.
    - Consider the function $V(x) = \frac{1}{2}(x_1^2 - x_2^2)$. $V(x) > 0$ on the line $x_2 = 0$.
    - The set U is set as before as shown in here.
$$\dot{V}(x) = x_1^2 + x_2^2 + x_1 g_1(x) - x_2 g_2(x)$$
    - while
$$|x_1 g_1(x) - x_2 g_2(x)| \le \sum_{i=1}^{2} |x_i| \cdot |g_i(x)| \le 2k\|x\|_2^3$$
    - Hence,
$$\dot{V}(x) \ge \|x\|_2^2 - 2k\|x\|_2^3 = \|x\|_2^2(1 - 2k\|x\|_2)$$
    - Choose $r < 1/(2k)$, Then $\dot{V}$ becomes positive, and the origin is unstable.

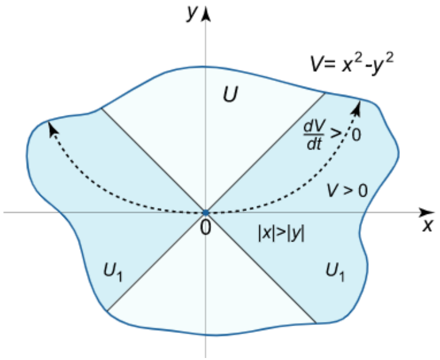
Scientist Bio

Nikolay Gur'yevich Chetaev

(23 November 1902 – 17 October 1959)

Was a Russian Soviet mechanician and mathematician. He is renowned from his work on elliptic_partial_differential_equations. He belongs to the Kazan school of mathematics. From 1930 to 1940 N. G. Chetaev was a professor of Kazan University where he created a scientific school of the mathematical theory of stability of motion, where N. Krosovskii was a Ph.D. student.

Chetaev made a number of significant contributions to Mathematical Theory of Stability, Analytical Mechanics and Mathematical Physics. His major scientific achievements relates to as: The Poincaré equations; Lagrange's theorem of stability of an equilibrium, Poincaré–Lyapunov theorem on a periodic motion & Chetaev's theorems; Chetaev's method of constructing Lyapunov functions as a coupling (combination) of first integrals; D'Alembert–Lagrange and Gauss Principles.



A representative region for Chetaev instability Theorem

Contents

In this chapter we review the general stability analysis of autonomous nonlinear system, through Lyapunov direct and indirect methods, and invariance principles. Furthermore, Lyapunov function generation and Lyapunov-based controller design are reviewed in detail. This chapter contains the most important analysis design of nonlinear systems.

Lyapunov Stability of Autonomous Systems

- Invariant Set Theorems:
  - ✓ Asymptotic stability needs $\dot{V}(x) < 0$
    - In many systems we may reach only to $\dot{V}(x) \leq 0$
    - Use invariant set to prove asymptotic stability
  - ✓ A set M is an invariant set with respect to (3.1) if

$$x(0) \in M \quad \Rightarrow \quad x(t) \in M, \qquad \forall t \in R$$

  - ✓ A set M is an positively invariant set with respect to (3.1) if

$$x(0) \in M \quad \Rightarrow \quad x(t) \in M, \qquad \forall t \geq 0$$

    - Examples of invariant sets
      - An equilibrium point
      - A limit cycle
      - Any trajectory
      - The RoA of an eq. point or a limit cycle
      - The whole state space

Invariant Set Theorem

- Barbashin - Krasovskii - Lasalle's Theorem:

Theorem 3.4 *Let $\Omega \subset D$ be a compact set that is positively invariant with respect to (3.1). Let $V : D \to R$ be a continuously differentiable function such that $\dot{V}(x) \le 0$ in Ω. Let E be the set of all points in Ω where $\dot{V}(x) = 0$. Let M be the largest invariant set in E. Then every solution starting in Ω approaches M as $t \to \infty$.* ◇

- It is not directly stated that $V(x) > 0$, But
- The function V is continuous on the compact set Ω
  - It is bounded from below (somehow positive)
  - The set Ω_l is called a compact set.
- Largest invariant set means the union of all invariant sets.
- This theorem introduces the notion of Region of Attraction.
- Can be used for Eq. point, limit cycle, or any invariant set.

Invariant Set Theorem

- Lasalle's Corollaries:
  - ✓ Local Asymptotic Stability

Corollary 3.1 *Let $x = 0$ be an equilibrium point for (3.1). Let $V : D \to R$ be a continuously differentiable positive definite function on a domain D containing the origin $x = 0$, such that $\dot{V}(x) \leq 0$ in D. Let $S = \{x \in D \mid \dot{V}(x) = 0\}$ and suppose that no solution can stay identically in S, other than the trivial solution. Then, the origin is asymptotically stable.* ◇

  - ✓ Global Asymptotic Stability

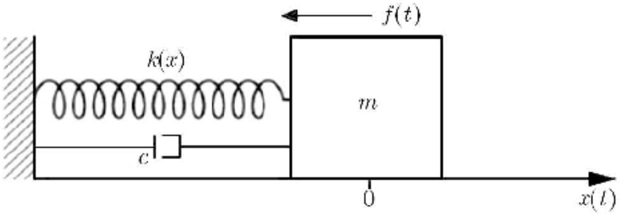
Corollary 3.2 *Let $x = 0$ be an equilibrium point for (3.1). Let $V : R^n \to R$ be a continuously differentiable, radially unbounded, positive definite function such that $\dot{V}(x) \leq 0$ for all $x \in R^n$. Let $S = \{x \in R^n \mid \dot{V}(x) = 0\}$ and suppose that no solution can stay identically in S, other than the trivial solution. Then, the origin is globally asymptotically stable.* ◇

Invariant Set Theorem

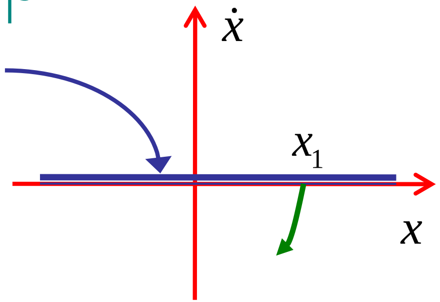
✓ Example: Mass with nonlinear spring-damper

- Consider the system:

$$m\ddot{x} + c\dot{x}|\dot{x}| + k_0 + k_1 x^3 = 0$$



- Lyapunov candidate
  - $V(x) = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} k_0 x^2 + \frac{1}{4} k_1 x^4$
  - $\dot{V}(x) = m\dot{x}\ddot{x} + (k_{0x} + k_1 x^3)\dot{x} = \dot{x}(-c\dot{x}|\dot{x}|) = -c|\dot{x}|^3$
  - The set R where $\dot{V}(x) = 0$ is $R = \{(x, \dot{x}) | \dot{x} = 0\}$
  - Is the largest invariant set in $R, M = \{(0,0)\}$?



Suppose any arbitrary point of R, such as $(x_1, 0)$ is also in M.

Any trajectory passing through this point must satisfy:

$\ddot{x} = -\left(\frac{k_0}{m}\right) x - \left(\frac{k_1}{m}\right) x^3 \neq 0$, hence, the trajectory moves out from R.

  - The equilibrium point is asymptotically stable.

Invariant Set Theorem

✓ Example 2:

- System dynamics:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -g(x_1) - h(x_2)$$

  – In which,

$$g(0) = 0, \quad yg(y) > 0, \quad \forall\, y \neq 0, \; y \in (-a, a)$$
$$h(0) = 0, \quad yh(y) > 0, \quad \forall\, y \neq 0, \; y \in (-a, a)$$

- Eq. point @ origin.
- Lyapunov Candidate:

$$V(x) = \int_0^{x_1} g(y)\, dy + \frac{1}{2} x_2^2$$

  – In the domain $D = \{x \in R^2 \mid -a < x_i < a\}$ is positive definite.

- Lyapunov function derivative:

$$\dot{V}(x) = g(x_1)x_2 + x_2[-g(x_1) - h(x_2)] = -x_2 h(x_2) \leq 0$$

  – Positive semi-definite, needs invariant set Theorem.

Invariant Set Theorem

✓ Example 2: (cont.)

- Characterize the Set R where:

$$\dot{V}(x) = 0 \;\Rightarrow\; x_2 h(x_2) = 0 \;\Rightarrow\; x_2 = 0, \;\; \text{since} \; -a < x_2 < a$$

- Hence, $R = \{(x_1, x_2) \mid x_2 = 0\}$
- Show that M includes only origin:
  - Suppose x(t) is a trajectory belonging to R, then

$$x_2(t) \equiv 0 \Rightarrow \dot{x}_2(t) \equiv 0 \Rightarrow g(x_1(t)) \equiv 0 \Rightarrow x_1(t) \equiv 0$$

  - Hence, the solution to this trajectory is only the origin.

- The equilibrium point in asymptotically stable.

Invariant Set Theorem

✓ Example 3: Region of Attraction

- System dynamics:

$$\dot{x}_1 = x_1(x_1^2 + x_2^2 - 2) - 4x_1 x_2^2$$

$$\dot{x}_2 = 4x_1^2 x_2 + x_2(x_1^2 + x_2^2 - 2)$$

- Eq. point is @ origin.
- Lyapunov candidate: $V(x_1, x_2) = x_1^2 + x_2^2$
  - for $l = 2$, the region Ω_l is defined by $V(x) < 2$ is a compact set.
- Derivative along trajectory: $\dot{V} = 2(x_1^2 + x_2^2)(x_1^2 + x_2^2 - 2)$
  - For the set Ω_l the derivative is always negative except @ origin.
- The set R includes only the origin.
  - Invariant Set Theorem conditions hold.
  - The eq. point is locally asymptotically stable.
  - The Region of Attraction is estimated by Ω_l a circle with radius $r = \sqrt{2}$.

Invariant Set Theorem

✓ Example 4: Attractive Limit Cycle

- System dynamics

$$\dot{x}_1 = x_2 - x_1^7 [\, x_1^4 + 2x_2^2 - 10 \,]$$

$$\dot{x}_2 = -x_1^3 - 3x_2^5 [\, x_1^4 + 2x_2^2 - 10 \,]$$

- There exist an invariant set: $x_1^4 + 2x_2^2 = 10$
  - Since, its derivative is zero on the set.

$$\frac{d}{dt}(x_1^4 + 2x_2^2 - 10) = -(4x_1^{10} + 12x_2^6)(x_1^4 + 2x_2^2 - 10) = 0$$

- On the invariant set:
  - Simplified system dynamics
  - Invariant set is a limit cycle

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1^3$$

- Is the limit cycle attractive?
  - Lyapunov candidate: $V = (x_1^4 + 2x_2^2 - 10)^2$
  - Physical insight: distance to the limit cycle.

Invariant Set Theorem

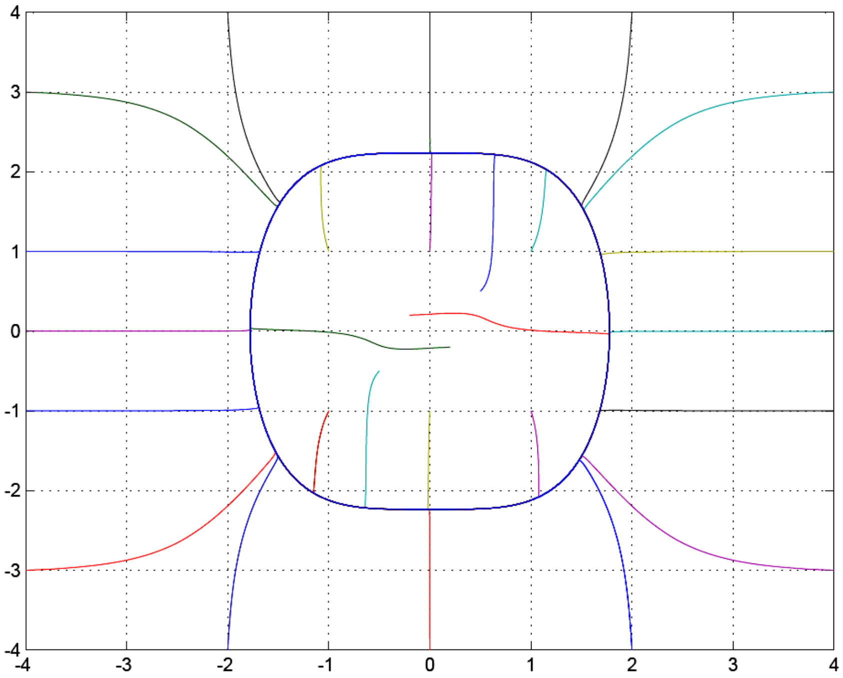
✓ Example 4: Attractive Limit Cycle (cont.)

- For any $l > 0$,
  - the region Ω_l defined by $V(x) < l$ is a compact set.
- Lyapunov function derivative:
  - from before, $\dot{V} = -8\,(x_1^{10} + 3\,x_2^{6})\,(x_1^{4} + 2\,x_2^{2} - 10)^2$
  - $\dot{V}(x) < 0$ everywhere except at
    - $x_1^4 + 2x_2^2 = 10$ The limit Cycle.
    - $x_1^{10} + 3x_2^6 = 0$ The Eq. point @ origin.
  - The eq. point at origin is unstable.
- From invariant set theorem, all the trajectories converge to the limit cycle.

Invariant Set Theorem

✓ Example 4: Attractive Limit Cycle (cont.)

- Phase portrait:



Scientist Bio

Nikolay Nikolayevich Krasovskii
(7 September 1924 – 4 April 2012)

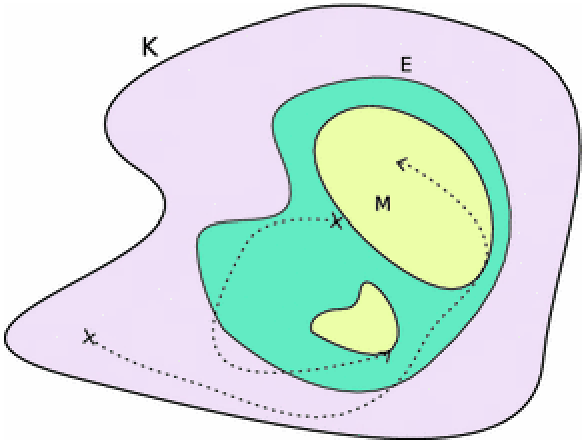
Was a prominent Russian mathematician who worked in the mathematical theory of control, the theory of dynamical systems, and the theory of differential games. He was the author of Krasovskii-LaSalle principle and the chief of the Ural scientific school in mathematical theory of control and the theory of differential games. In 1963 Stanford University Press published a translation of his book *Stability of Motion: applications of Lyapunov's second method to differential systems and equations with delay* that had been prepared by Joel Lee Brenner. Krasovskii received many honours for his contributions. He was elected a corresponding member of the USSR Academy of Sciences in 1964 and became a full member in 1968. He was awarded the M V Lomonosov Gold Medal of the Russian Academy of Sciences, the A M Lyapunov Gold Medal, the Demidov Prize in physics and mathematics, and the 'Triumph' Prize which is awarded to the leading scientists for their contribution to Russian and world science as a whole.

Scientist Bio

Joseph P. LaSalle

(28 May 1916 - 7 July 1983)

Was an American mathematician, specialising in dynamical systems and responsible for important contributions to stability theory, such as LaSalle's invariance principle which bears his name. Joseph LaSalle defended his Ph.D. thesis on "*Pseudo-Normed Linear Sets over Valued Rings*" at the California Institute of Technology in 1941. During a visit to Princeton in 1947–1948, LaSalle developed a deep interest in differential equations through his interaction with Solomon Lefschetz and Richard Bellman. he worked closely with Lefschetz and in 1960 published his extension of Lyapunov stability theory,[5] known today as LaSalle's invariance principle.



For the proof of LaSalle's invariance principle

Contents

In this chapter we review the general stability analysis of autonomous nonlinear system, through Lyapunov direct and indirect methods, and invariance principles. Furthermore, Lyapunov function generation and Lyapunov-based controller design are reviewed in detail. This chapter contains the most important analysis design of nonlinear systems.

Lyapunov Stability of Autonomous Systems

- ## Linear Systems and Linearization
  - ✓ Consider the linear time invariant system

$$\dot{x} = Ax \qquad (3.9)$$

    - That has an eq. point @ origin.
    - The eq. point is isolated if $\det(A) \neq 0$.
    - The solution for a given initial state $x(0)$:

$$x(t) = \exp(At)\, x(0)$$

Theorem 3.5 *The equilibrium point $x = 0$ of (3.9) is stable if and only if all eigenvalues of A satisfy $Re\lambda_i \leq 0$ and every eigenvalue with $Re\lambda_i = 0$ has an associated Jordan block of order one. The equilibrium point $x = 0$ is (globally) asymptotically stable if and only if all eigenvalues of A satisfy $Re\lambda_i < 0$.* ◇

Linear Systems and Linearization

- Stability of LTI system
  - ✓ When all eigenvalues of A satisfy $\mathrm{Re}\lambda_i < 0$,
    - A is called Hurwitz matrix.
  - ✓ For stability analysis consider $V(x) = x^T P x$
    - Where P is a real, symmetric and positive definite matrix.
    - Then

$$\dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x = x^T (PA + A^T P) x = -x^T Q x$$

    - Where Q is a positive definite matrix defined by

$$PA + A^T P = -Q \tag{3.12}$$

    - This is called Lyapunov Equation.

Linear Systems and Linearization

- Stability of LTI system

Theorem 3.6 *A matrix A is a stability matrix; that is, $Re\lambda_i < 0$ for all eigenvalues of A, if and only if for any given positive definite symmetric matrix Q there exists a positive definite symmetric matrix P that satisfies the Lyapunov equation (3.12). Moreover, if A is a stability matrix, then P is the unique solution of (3.12).* ◇

✓ Proof:

- Sufficiency follows from Theorem 3.1
- For Necessity assume all eigenvalues of A satisfy $\mathrm{Re}\lambda_i < 0$, and

$$P = \int_0^\infty \exp(A^T t) Q \exp(At)\ dt \tag{3.13}$$

- P exists and is proved to be positive definite by contradiction :

$$x^T P x = 0 \quad \Rightarrow \quad \int_0^\infty x^T \exp(A^T t) Q \exp(At) x\ dt = 0$$
$$\Rightarrow \quad \exp(At)x \equiv 0,\ \forall\ t \geq 0 \quad \Rightarrow \quad x = 0$$

Linear Systems and Linearization

- Stability of LTI system
  - ✓ Proof: (Cont.)
    - Since $\exp(At)$ is nonsingular, this results in contradiction.
      - Substitute (3.13) in (3.12)

$$
\begin{aligned}
PA + A^T P &= \int_0^\infty \exp(A^T t) Q \exp(At) A \, dt + \int_0^\infty A^T \exp(A^T t) Q \exp(At) \, dt \\
&= \int_0^\infty \frac{d}{dt} \exp(A^T t) Q \exp(At) \, dt = \exp(A^T t) Q \exp(At) \Big|_0^\infty = -Q
\end{aligned}
$$

      - This shows that P is a solution of (3.12).
      - Prove it is a unique solution by contradiction $(P - \tilde{P})A + A^T(P - \tilde{P}) = 0$
      - Pre- post multiply by $\exp(A^T t)$

$$0 = \exp(A^T t)[(P - \tilde{P})A + A^T(P - \tilde{P})] \exp(At) = \frac{d}{dt} \exp(A^T t)(P - \tilde{P}) \exp(At)$$

      - Hence,

$$\exp(A^T t)(P - \tilde{P}) \exp(At) \equiv \text{a constant } \forall\, t$$

      - Since, $\exp(A0) = \mathrm{I}$:

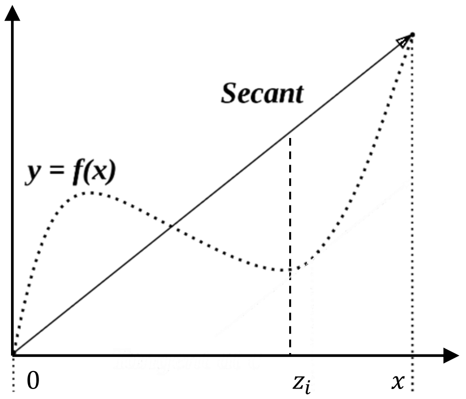
$$(P - \tilde{P}) = \exp(A^T t)(P - \tilde{P}) \exp(At) \to 0 \text{ as } t \to \infty$$

      - Therefore, $\tilde{P} = P$.

Linear Systems and Linearization

- Linearization of nonlinear system
  - ✓ Consider $\dot{x} = f(x)$, and $(0,0)$ its eq. point.
    - Let $f: D \rightarrow R^n$ be a continuously differentiable vector field
    - By mean value theorem

$$f_i(x) = f_i(0) + \frac{\partial f_i}{\partial x}(z_i)\ x$$



- Where z_i lies on the line segment from 0 to x
- Since $f_i(0) = 0$, write

$$f_i(x) = \frac{\partial f_i}{\partial x}(z_i)x = \frac{\partial f_i}{\partial x}(0)x + \left[\frac{\partial f_i}{\partial x}(z_i) - \frac{\partial f_i}{\partial x}(0)\right] x$$

$$f(x) = Ax + g(x)$$

$$A = \frac{\partial f}{\partial x}(0), \quad \text{and} \quad g_i(x) = \left[\frac{\partial f_i}{\partial x}(z_i) - \frac{\partial f_i}{\partial x}(0)\right] x$$

Linear Systems and Linearization

- Linearization of nonlinear system
  - The function $g_i(x)$ satisfies:

$$|g_i(x)| \le \left\| \frac{\partial f_i}{\partial x}(z_i) - \frac{\partial f_i}{\partial x}(0) \right\| \|x\|$$

  - By continuity

$$\frac{\|g(x)\|}{\|x\|} \to 0 \ \text{ as } \ \|x\| \to 0$$

  - This suggests that in a small neighborhood of the origin we can approximate $\dot{x} = f(x)$ by its linearization:

$$\dot{x} = Ax, \quad \text{where} \quad A = \frac{\partial f}{\partial x}(0)$$

  - How about the stability of the origin? Any Condition?

Linear Systems and Linearization

- Stability Analysis by Linearization
  - Lyapunov indirect or linearization method

Theorem 3.7 *Let $x = 0$ be an equilibrium point for the nonlinear system*

$$\dot{x} = f(x)$$

where $f : D \rightarrow R^n$ is continuously differentiable and D is a neighborhood of the origin. Let

$$A = \frac{\partial f}{\partial x}(x)\bigg|_{x=0}$$

Then,

1. *The origin is asymptotically stable if $Re\lambda_i < 0$ for all eigenvalues of A.*
2. *The origin is unstable if $Re\lambda_i > 0$ for one or more of the eigenvalues of A.*

- If eq. point is non-hyperbolic → Inconclusive!

Linear Systems and Linearization

- Stability Analysis by Linearization
  - ✓ Example 1:
    - Consider the system $\dot{x} = ax^3$

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = 3ax^2\big|_{x=0} = 0$$

      – The eigenvalue is on imaginary axis → Inconclusive!

    - Example 2: Pendulum

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\left(\frac{g}{l}\right)\sin x_1 - \left(\frac{k}{m}\right)x_2 \end{aligned}$$

      – The eq. points are @ (0,0) and (π,0).
      – Jacobian:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\left(\frac{g}{l}\right)\cos x_1 & -\left(\frac{k}{m}\right) \end{bmatrix}$$

Linear Systems and Linearization

- Stability Analysis by Linearization
  - Example 2: Pendulum (cont.)
    - For $(0,0)$ Eq. point :

$$A = \frac{\partial f}{\partial x}\Big|_{x=0} = \begin{bmatrix} 0 & 1 \\ -\left(\frac{g}{l}\right) & -\left(\frac{k}{m}\right) \end{bmatrix} \longrightarrow \lambda_{1,2} = \boxed{-\frac{k}{2m}} \pm \frac{1}{2}\sqrt{\left(\frac{k}{m}\right)^2 - \frac{4g}{l}}$$

      - All Eigenvalues Hurwitz → Asymptotically Stable
    - For $(\pi, 0)$ eq. point.
      - Change variables $z_1 = x_1 - \pi, z_2 = x_2$
      - Chack Jacobian @ $z = 0$

$$\tilde{A} = \frac{\partial f}{\partial x}\Big|_{x_1=\pi, x_2=0} = \begin{bmatrix} 0 & 1 \\ \left(\frac{g}{l}\right) & -\left(\frac{k}{m}\right) \end{bmatrix} \longrightarrow \lambda_{1,2} = \boxed{-\frac{k}{2m}} \pm \frac{1}{2}\sqrt{\left(\frac{k}{m}\right)^2 + \frac{4g}{l}}$$

      - One of the eigenvalues is not Hurwitz → Unstable

Contents

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Lyapunov Stability of Autonomous Systems

- Lyapunov Function Generation
  - ✓ Krasovskii Method

Theorem: **(Krasovskii)** *Consider the autonomous system defined by* (3.1), *with the equilibrium point of interest being the origin. Let* $\mathbf{A}(\mathbf{x})$ *denote the Jacobian matrix of the system, i.e.,*

$$\mathbf{A}(\mathbf{x}) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}$$

If the matrix $\mathbf{F} = \mathbf{A} + \mathbf{A}^T$ *is negative definite in a neighborhood* Ω, *then the equilibrium point at the origin is asymptotically stable. A Lyapunov function for this system is*

$$V(\mathbf{x}) = \mathbf{f}^T(\mathbf{x})\,\mathbf{f}(\mathbf{x})$$

If Ω *is the entire state space and, in addition,* $V(\mathbf{x}) \to \infty$ *as* $\|\mathbf{x}\| \to \infty$, *then the equilibrium point is globally asymptotically stable.*

Lyapunov Function Generation

- Krasovskii Method
  - Example: Consider the system

$$\dot{x}_1 = -6x_1 + 2x_2$$
$$\dot{x}_2 = 2x_1 - 6x_2 - 2x_2^3$$

    – The Jacobian:

$$\mathbf{A} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{bmatrix} -6 & 2 \\ 2 & -6-6x_2^2 \end{bmatrix} \qquad \mathbf{F} = \mathbf{A} + \mathbf{A}^T = \begin{bmatrix} -12 & 4 \\ 4 & -12-12x_2^2 \end{bmatrix}$$

    – F is negative definite for the whole space.
    – Lyapunov Function $V(\mathbf{x}) = \mathbf{f}^T(\mathbf{x})\,\mathbf{f}(\mathbf{x}) = (-6x_1 + 2x_2)^2 + (2x_1 - 6x_2 - 2x_2^3)^2$
    – It is Radially unbounded

$$V(\mathbf{x}) \to \infty \quad \text{as} \quad \|\mathbf{x}\| \to \infty$$

    – The Eq. point is globally asymptotically stable.

Lyapunov Function Generation

- Lyapunov Function Generation
  - ✓ Krasovskii Method

Theorem 3.8 (Generalized Krasovskii Theorem) *Consider the autonomous system defined by (3.2), with the equilibrium point of interest being the origin, and let* $\mathbf{A}(\mathbf{x})$ *denote the Jacobian matrix of the system. Then, a sufficient condition for the origin to be asymptotically stable is that there exist two symmetric positive definite matrices* $\mathbf{P}$ *and* $\mathbf{Q}$*, such that* $\forall \mathbf{x} \neq \mathbf{0}$*, the matrix*

$$\mathbf{F}(\mathbf{x}) = \mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{Q}$$

is negative semi-definite in some neighborhood $\mathbf{\Omega}$ *of the origin. The function* $V(\mathbf{x}) = \mathbf{f}^T\mathbf{P}\mathbf{f}$ *is then a Lyapunov function for the system. If the region* $\mathbf{\Omega}$ *is the whole state space, and if in addition,* $V(\mathbf{x}) \rightarrow \infty$ *as* $\|\mathbf{x}\| \rightarrow \infty$*, then the system is globally asymptotically stable.*

- Proof Idea: $\dot{V} = \dfrac{\partial V}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}) = \mathbf{f}^T\mathbf{P}\mathbf{A}(\mathbf{x})\mathbf{f} + \mathbf{f}^T\mathbf{P}\mathbf{A}^T(\mathbf{x})\mathbf{P}\mathbf{f} = \mathbf{f}^T\mathbf{F}\mathbf{f} - \mathbf{f}^T\mathbf{Q}\mathbf{f}$
  - If $F < 0$ and $Q > 0$, then $\dot{V}(x) < 0$.

Lyapunov Function Generation

- Variable Gradient Method
  - ✓ Search backward, start with $\dot{V}(x) < 0$, then find $V(x)$.
    - Procedure:
      - Suppose $g(x)$ is the gradient of $V(x)$:
      - Derivative of $V(x)$ along trajectory: $g(x) = \nabla V = (\partial V / \partial x)^T$

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) = g^T(x) f(x)$$

      - Choose $g(x)$ such that $\dot{V}(x) < 0$, while $V(x) > 0$.
      - For g(x) to be gradient of a scalar function:

$$\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}, \quad \forall\, i, j = 1, \ldots, n$$

      - Under this constraint choose $g(x)$ such that $g^T(x) f(x) < 0$

Lyapunov Function Generation

- Variable Gradient Method
  - Procedure (cont.):
    - Then generate $V(x)$ by integration

$$V(x) = \int_0^x g^T(y)\, dy = \int_0^x \sum_{i=1}^n g_i(y)\, dy_i$$

    - The integration can be taken along any path, but usually it is taken along the principal axes:

$$\begin{aligned} V(x) &= \int_0^{x_1} g_1(y_1, 0, \ldots, 0)\, dy_1 + \int_0^{x_2} g_2(x_1, y_2, 0, \ldots, 0)\, dy_2 \\ &+ \cdots + \int_0^{x_n} g_n(x_1, x_2, \ldots, x_{n-1}, y_n)\, dy_n \end{aligned}$$

    - Leave some parameters of $g(x)$ undetermined, and try to choose them to ensure that $V(x)$ positive.

Lyapunov Function Generation

- Variable Gradient Method
  - ✓ Example 1:
    - Consider the system:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -h(x_1) - a x_2\end{aligned}$$

      – where, $a > 0, h(0) = 0$ and $y\,h(y) > 0, \forall y \in (-b, c).$
    - To ensure $\dot{V}(x) < 0 \rightarrow g^T(x) f(x) < 0$

$$\dot{V}(x) = g_1(x)x_2 - g_2(x)[h(x_1) + a x_2] < 0, \quad \text{for } x \neq 0$$

    - The Lypunov function is:

$$V(x) = \int_0^x g^T(y)\,dy > 0, \quad \text{for } x \neq 0$$

    - Let us try

$$g(x) = \begin{bmatrix} \alpha(x)x_1 + \beta(x)x_2 \\ \gamma(x)x_1 + \delta(x)x_2 \end{bmatrix}$$

    - Gradient condition

$$\frac{\partial g_1}{\partial x_2} = \frac{\partial g_2}{\partial x_1} \longrightarrow \beta(x) + \frac{\partial \alpha}{\partial x_2}x_1 + \frac{\partial \beta}{\partial x_2}x_2 = \gamma(x) + \frac{\partial \gamma}{\partial x_1}x_1 + \frac{\partial \delta}{\partial x_1}x_2$$

Lyapunov Function Generation

- Variable Gradient Method
  - ✓ Example 1: (cont.)
    - Derivative of Ly. f.

$$\begin{aligned}\dot V(x) &= \alpha(x)x_1x_2 + \beta(x)x_2^2 - a\gamma(x)x_1x_2 \\ &\quad -a\delta(x)x_2^2 - \delta(x)x_2h(x_1) - \gamma(x)x_1h(x_1)\end{aligned}$$

      - To cancel cross terms $\alpha(x)x_1 - a\gamma(x)x_1 - \delta(x)h(x_1) = 0$
      - Therefore,

$$\dot V(x) = -[a\delta(x) - \beta(x)]x_2^2 - \gamma(x)x_1h(x_1)$$

      - To simplify assign $\beta, \gamma,$ and δ to be constant but keep $\alpha(x)$
    - From gradient condition

$$\beta(x) + \frac{\partial\alpha}{\partial x_2}x_1 + \frac{\partial\beta}{\partial x_2}x_2 = \gamma(x) + \frac{\partial\gamma}{\partial x_1}x_1 + \frac{\partial\delta}{\partial x_1}x_2$$

      - $\alpha(x) = \alpha(x_1)$ and $\beta = \gamma$.
      - $$g(x) = \begin{bmatrix}\alpha(x)x_1 + \beta(x)x_2 \\ \gamma(x)x_1 + \delta(x)x_2\end{bmatrix} \Rightarrow g(x) = \begin{bmatrix} a\gamma x_1 + \delta h(x_1) + \gamma x_2 \\ \gamma x_1 + \delta x_2\end{bmatrix}$$

Lyapunov Function Generation

- Variable Gradient Method
  - ✓ Example 1: (cont.)
    - Integrate g(x) to get the Ly. function

$$V(x) = \int_0^{x_1} [a\gamma y_1 + \delta h(y_1)]\, dy_1 + \int_0^{x_2} (\gamma x_1 + \delta y_2)\, dy_2$$

$$= \tfrac{1}{2}a\gamma x_1^2 + \delta \int_0^{x_1} h(y)\, dy + \gamma x_1 x_2 + \tfrac{1}{2}\delta x_2^2 = \tfrac{1}{2}x^T P x + \delta \int_0^{x_1} h(y)\, dy$$

– in which, $P = \begin{bmatrix} a\gamma & \gamma \\ \gamma & \delta \end{bmatrix}$

- Choose $\delta > 0$, and $0 < \gamma < a\delta$ to ensure $V(x) > 0$, and $\dot{V}(x) < 0$
  - For example $\gamma = ak\delta$ for $0 < k < 1$
  - Then, $V(x) = \dfrac{\delta}{2} x^T \begin{bmatrix} ka^2 & ka \\ ka & 1 \end{bmatrix} x + \delta \int_0^{x_1} h(y)\, dy$
  - $V(x) > 0$, and $\dot{V}(x) < 0$ for $D = \{x \in R^2 | -b < x_1 < c\}$
  - The eq. point is asymptotically stable.

Contents

In this chapter we review the general stability analysis of autonomous nonlinear system, through Lyapunov direct and indirect methods, and invariance principles. Furthermore, Lyapunov function generation and Lyapunov-based controller design are reviewed in detail. This chapter contains the most important analysis design of nonlinear systems.

Lyapunov Stability of Autonomous Systems

- Lyapunov Based Controller Design
  - ✓ Example: Robotic Manipulator
    - Physically derived Lyapunov function
    - System dynamics $$\mathbf{H}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{b}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau}$$
    - Controller $$\boldsymbol{\tau} = -\mathbf{K}_D\dot{\mathbf{q}} - \mathbf{K}_P\mathbf{q} + \mathbf{g}(\mathbf{q})$$
    - Lyapunov Candidate $$V = \frac{1}{2}[\dot{\mathbf{q}}^T\mathbf{H}\dot{\mathbf{q}} + \mathbf{q}^T\mathbf{K}_p\mathbf{q}]$$
      - Total Energy
    - Lyapunov Function Derivative
      - Power of the external forces $$\dot{V} = \dot{\mathbf{q}}^T(\boldsymbol{\tau} - \mathbf{g}) + \dot{\mathbf{q}}^T\mathbf{K}_P\mathbf{q}$$
      - Used control law $$\dot{V} = -\dot{\mathbf{q}}^T\mathbf{K}_D\dot{\mathbf{q}}$$
      - Lasalle: Global Asymptotically stable

Lyapunov Stability of Autonomous Systems

- Lyapunov Based Controller Design
  - ✓ Design Idea:
    - Consider a Lyapunov candidate
    - Stability:
      - – Design the control law as a nonlinear function to ensure negative definiteness of the Ly. F. Derivative.
    - Performance:
      - – Rate of decay is related to the time performance.
    - Base of many nonlinear controller designs:
      - – Back-stepping
      - – Sliding mode control
      - – Lyapunov redesign
      - – . . .

Lyapunov Stability of Autonomous Systems

- Lyapunov Based Controller Design
  - ✓ Example 2: Regulation
    - System dynamics: $\ddot{x} - \dot{x}^3 + x^2 = u$
    - Objective
      - Push the trajectories toward origin.
    - Consider the controller as: $u = u(x, \dot{x})$
    - Lyapunov Candidate: $V = 1/2(x^2 + \dot{x}^2)$
    - Derivative: $\dot{V} = \dot{x}(x + \dot{x}^3 - x^2 + u)$
    - Design u such that $\dot{V}(x) \le 0$: $\dot{V} \equiv -K\,\dot{x}^2 \rightarrow u = -x + x^2 - K\,\dot{x} - \dot{x}^3$
      - For example,
      - Stability: Lasalle → asymptotically stable eq. point.
      - Performance: increase K to have faster response.
      - Controller is not unique.

Nonlinear Control

Chapter 3: Lyapunov Stability of Autonomous Systems

To read more and see the course videos
visit our course website:
http://aras.kntu.ac.ir/arascourses/nonlinear-control/

Thank You

ARAS HI-TECH ROBOTIC SOLUTIONS
ADVANCED ROBOTICS & AUTOMATED SYSTEMS

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